Designing Fixed Field Accelerators from their Orbits

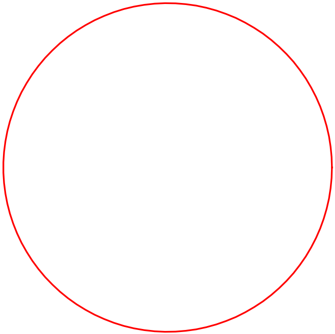
Thomas Planche

TRIUMF

Contents

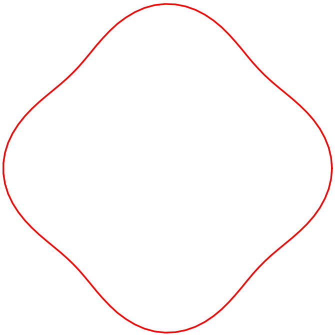
Ingredients

Circular closed orbit $r(\theta) = a$



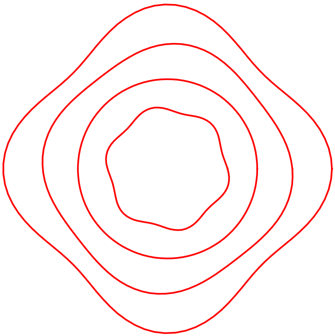
Ingredients

General closed orbit (Fourier series) $r(\theta) = \sum_{i=0}^{\infty} C_i \cos(i(\theta + \phi_i))$



Ingredients

Continuum of closed orbits: $r(\theta, a) = a \sum_{i=0}^{\infty} C_i(a) \cos(i(\theta + \phi_i(a)))$



a is the average radius of the orbit.
Note that $\frac{\partial r}{\partial a} > 0$ for orbits to not cross over.

Ingredients

Must assume some relation between orbit scale and momentum:

$$P(a)$$

Objective

Given $r(\theta, a)$ and $P(a)$, calculate the transverse tunes:

$$\nu_r(a)$$

$$\nu_z(a)$$

Linear Motion around Closed orbit [Courant and Snyder, 1958]

ANNALS OF PHYSICS: **3,** 1–48 (1958)

Theory of the Alternating-Gradient Synchrotron*†

E. D. COURANT AND H. S. SNYDER

Brookhaven National Laboratory, Upton, New York

The equations of motion of the particles in a synchrotron in which the field gradient index

$$n = -(r/B)\partial B/\partial r$$

varies along the equilibrium orbit are examined on the basis of the linear

Linear Motion around Closed orbit [Courant and Snyder, 1958]

The equations of motion are derived from the Hamiltonian

$$H = eV + c[m^2c^2 + (\mathbf{p} - e\mathbf{A})^2]^{1/2}, \tag{B9}$$

where V and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field. In terms of the new variables this equals

$$\begin{aligned} H = eV + c\Big\{m^2c^2 + \frac{1}{(1+\Omega x)^2}[p_s - eA_s + \omega z(p_x - eA_x) \\ - \omega x(p_z - eA_z)]^2 + (p_x - eA_x)^2 + (p_z - eA_z)^2\Big\}^{1/2}, \end{aligned} \tag{B10}$$

...

The linearized equations of motion are obtained by expanding G as a power series in x, p_x, z, p_z and retaining only terms up to the second order. We consider a static magnetic field, so that $V = 0$ and $\mathbf{A}$ is independent of time. We may choose a gauge such that the power series expansions of the components of $\mathbf{A}$ are in the form

$$\begin{aligned} A_s &= ax + bz + cx^2 + dxz + ez^2 + \cdots, \\ A_x &= -fz + \cdots, \\ A_z &= fx + \cdots. \end{aligned} \tag{B15}$$

Linear Motion Hamiltonian

Frenet-Serret coordinates (x, y, s):

$$(\nabla \times \mathbf{A})(0,0,s) = \begin{pmatrix} 0 \\ B_0(s) \\ 0 \end{pmatrix},$$

where $B_0(s) = B(0,0,s)$. The vector potential should also satisfy the absence of source along the orbit, which is:

$$(\nabla \times \nabla \times \mathbf{A})(0,0,s) = \mathbf{0}\,.$$

Linear Motion Hamiltonian

Following Courant and Snyder we find a suitable polynomial expansion:

$$\begin{aligned} A_x &= 0\,, \\ A_y &= \frac{\partial B(s)}{\partial s} xy\,, \\ A_s &= -\frac{B(s)}{2\rho(s)}\left(x^2(1+n(s)) + y^2 n(s)\right) - xB(s)\,, \end{aligned}$$

where $n = -\frac{\rho}{B_0}\left.\frac{\partial B}{\partial x}\right|_{x=y=0}$

Linear Motion Hamiltonian [Baartman, 2005]

Which leads after a canonical transformation $(t, -E) \to (z, \Delta P)$ using the generating function: $F_2(t, \Delta P) = \left(\frac{s}{\beta c} - t\right)(E_0 + \beta c \Delta P)$

$$h(x, p_x, y, p_y, z, \Delta P; s) = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{p_z x}{\rho} + \frac{p_z^2}{2\gamma^2}$$

where:

$$\rho = \frac{P}{qB_0} \, ,$$

$$n = -\frac{\rho}{B_0} \left.\frac{\partial B}{\partial x}\right|_{x=y=0} \, ,$$

$$p_x = P_x/P \, ,$$

$$p_y = P_y/P \, ,$$

$$p_z = \Delta P/P \, ,$$

$$h = H/P \, ,$$

and γ is the Lorentz factor

Linear Motion Hamiltonian [Baartman, 2005]

$$h = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{p_z x}{\rho} + \frac{p_z^2}{2\gamma^2}$$

Linear Motion Hamiltonian

$$h = \frac{x^2}{2}\frac{1-n}{\rho^2} + \frac{y^2}{2}\frac{n}{\rho^2} + \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{p_z x}{\rho} + \frac{p_z^2}{2\gamma^2}$$

ρ, γ, and n from geometry

Remember:

$$r(\theta, a) = a \sum_{i=0}^{\infty} C_i(a) \cos(i(\theta + \phi_i(a)))$$

From this we get:

$$\rho(a, \theta) = \frac{\left(r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2\right)^{3/2}}{r^2 + 2\left(\frac{\partial r}{\partial \theta}\right)^2 - r\frac{\partial^2 r}{\partial \theta^2}}$$

see en.wikipedia.org/wiki/Curvature

ρ, γ, and n from geometry

$\gamma(a)$ is up to you...

Isochronous cyclotron: $\beta = \frac{\mathcal{L}(a)}{r_\infty}$

with $\mathcal{L}(a) = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} \, \mathrm{d}\theta$

and of course: $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

ρ, γ, and n from geometry

$\gamma(a)$ is up to you…

Later in this presentation I will also use: $P(a) = P_0 \left(\frac{a}{a_0}\right)^{k+1}$

ρ, γ, and n from geometry

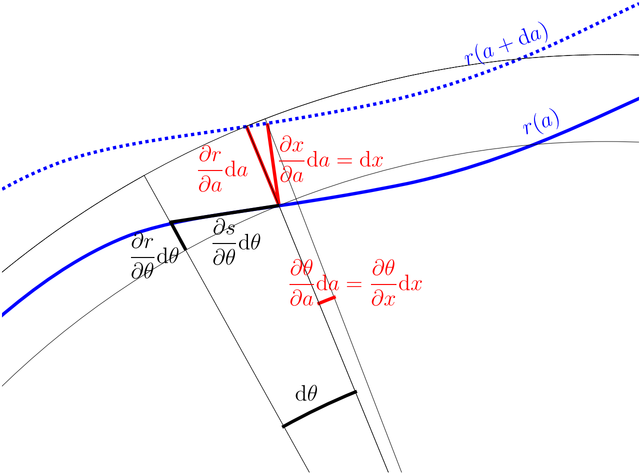
Chain $\rho = \frac{P}{qB_0}$ and chain rule:

$$n = -\frac{q\rho^2}{P}\frac{\partial B}{\partial x} = \frac{\partial \rho}{\partial x} - \frac{\rho}{P}\frac{\partial P}{\partial x}\,,$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial \rho}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{1}{r}\left(\frac{\partial \rho}{\partial a}\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{\frac{\partial r}{\partial a}} - \frac{\partial \rho}{\partial \theta}\frac{\frac{\partial r}{\partial \theta}}{\frac{\mathrm{d}s}{\mathrm{d}\theta}}\right)\,,$$

$$\frac{\partial P}{\partial x} = \frac{\mathrm{d}\beta}{\mathrm{d}a}\frac{mc}{r\left(1-\beta^2\right)^{3/2}}\frac{\frac{\mathrm{d}s}{\mathrm{d}\theta}}{\frac{\partial r}{\partial a}}\,.$$

ρ, γ, and n from geometry



Note that $\frac{\partial r}{\partial a} > 0$ for orbits to not cross over.

ν_r, and ν_z [Meade, 1971]

Infinitesimal matrix (terms are first derivatives of the Hamiltonian):

$$\mathbf{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{n(s)-1}{\rho(s)^2} & 0 & 0 & 0 & 0 & \frac{1}{\rho(s)} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{n(s)}{\rho(s)^2} & 0 & 0 & 0 \\ -\frac{1}{\rho(s)} & 0 & 0 & 0 & 0 & \frac{1}{\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that we know ρ, γ, and n, tunes are calculated by integrating:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\theta} = \mathbf{X}' \frac{\mathrm{d}s}{\mathrm{d}\theta} = \mathbf{F}\mathbf{X}\frac{\mathrm{d}s}{\mathrm{d}\theta}$$

over one period for two different sets of initial transverse state vectors: $\mathbf{X} = (1, 0, 1, 0, 0, 0)^\mathsf{T}$ and $\mathbf{X} = (0, 1, 0, 1, 0, 0)^\mathsf{T}$.

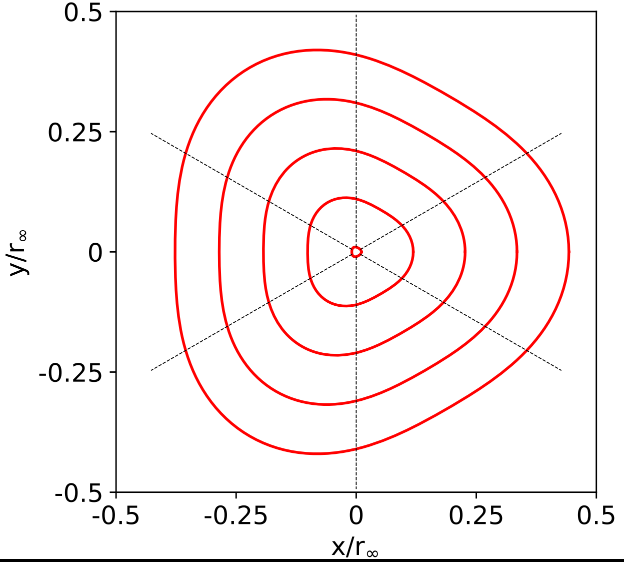
Example: Gordon's Cyclotron [Gordon, 1968]

Soft-edge version, with only 2 Fourier Harmonics:

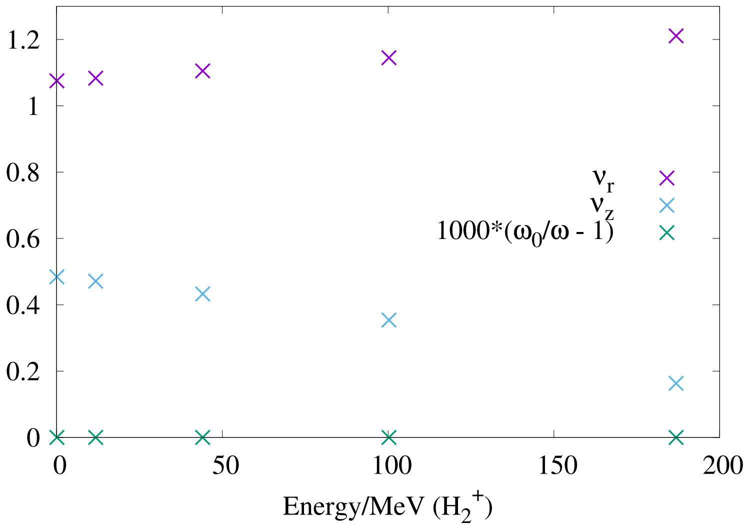
$$r(a, \theta) = a \left(1 + C \cos \left(N\theta\right)\right)$$

I choose the number of sectors $N = 3$.

Example: Gordon's Cyclotron [Gordon, 1968]



Example: Gordon's Cyclotron [Gordon, 1968]

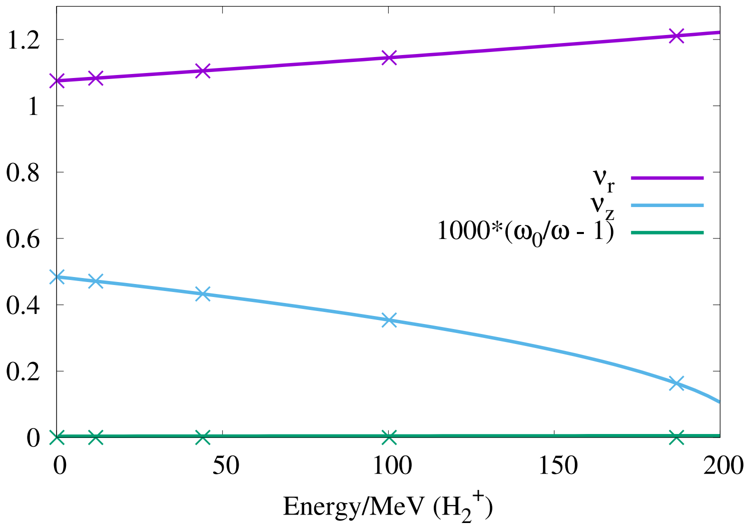


Verification: CYCLOPS

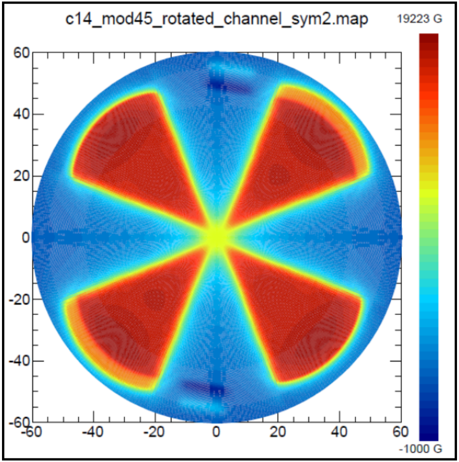
Polar field map:

$$B(r,\theta) = \frac{P(a(r,\theta))}{q\,\rho(r,\theta)}$$

Verification: CYCLOPS



Central Orbit: Circular!

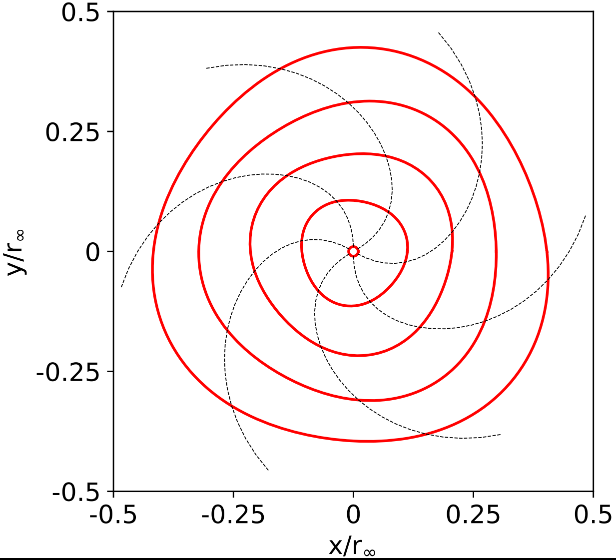


Courtesy of Wiel Kleeven, IBA, see talk THD03 in CYC'19 conference.

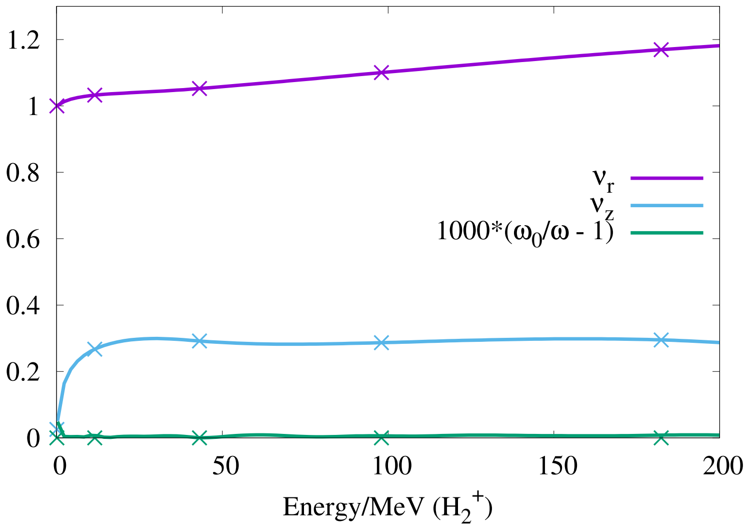
Example: Spiral Sector Cyclotron

$$r(a,\theta) = a\left(1 + C(a)\cos\left(N(\theta - \phi(a))\right)\right)$$

Example: Spiral Sector Cyclotron



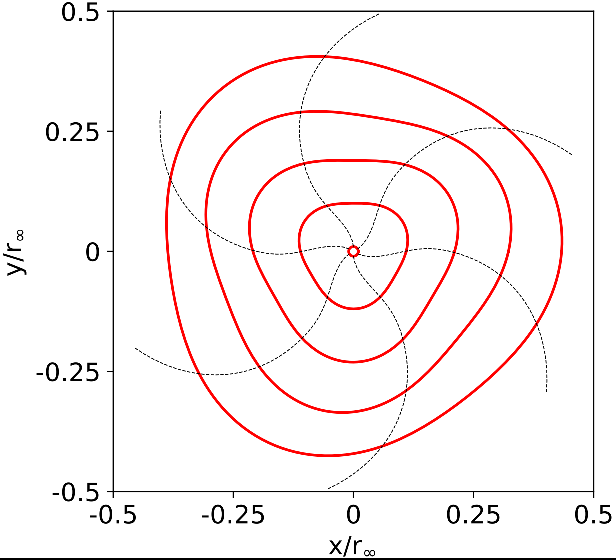
Example: Spiral Sector Cyclotron



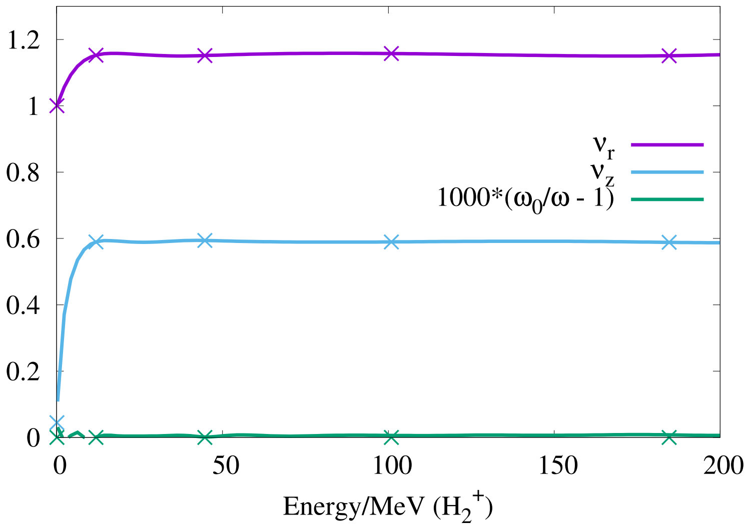
Example: Flat Tunes Cyclotron

$$r(a, \theta) = a\left(1 + C(a)\cos\left(N(\theta - \phi(a))\right)\right)$$

Example: Flat Tunes Cyclotron



Example: Flat Tunes Cyclotron

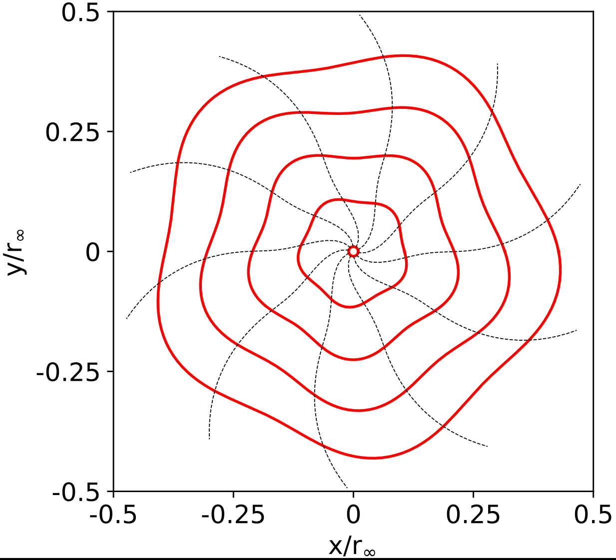


Example: 5 Sector Cyclotron

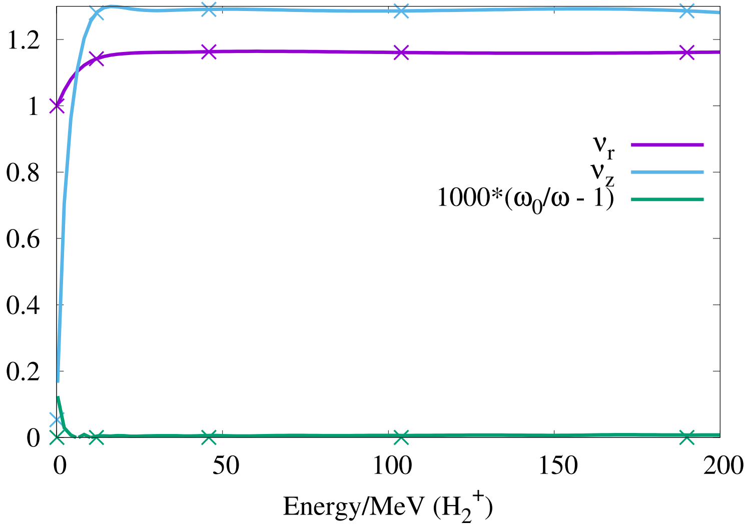
$$r(a,\theta) = a\left(1 + C(a)\cos\left(N(\theta - \phi(a))\right)\right)$$

with $N = 5$

Example: 5 Sector Cyclotron



Example: 5 Sector Cyclotron

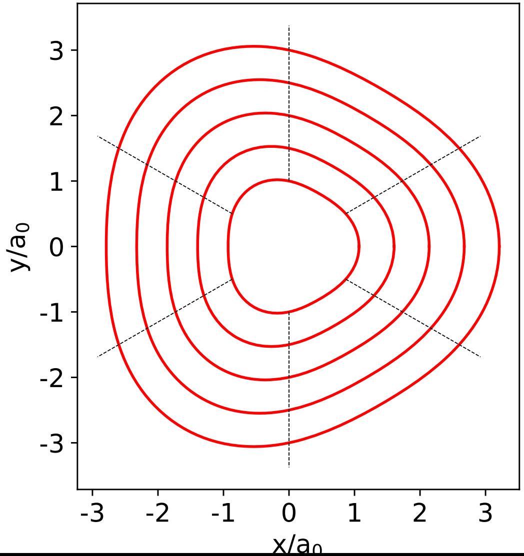


Example: Radial Sector Scaling FFA

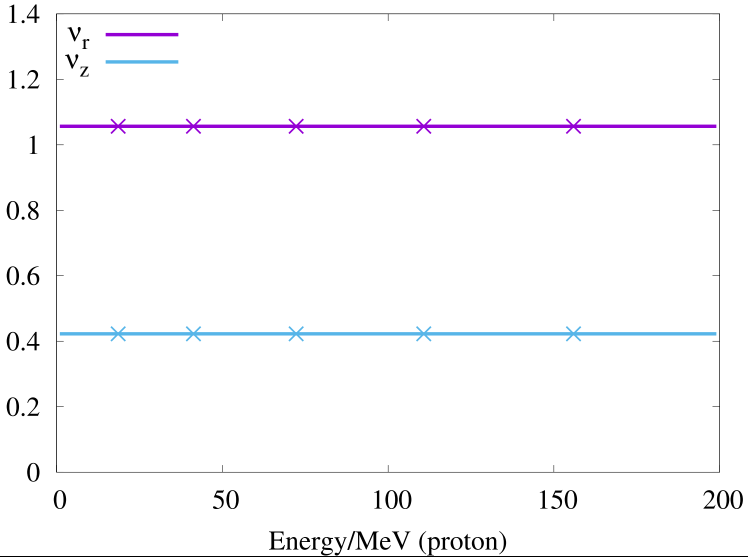
$$P(a) = P_0 \frac{a}{a_0}$$

$$r(a, \theta) = a \left(1 + C \cos\left(N\theta\right)\right)$$

Example: Radial Sector Scaling FFA



Example: Radial Sector Scaling FFA

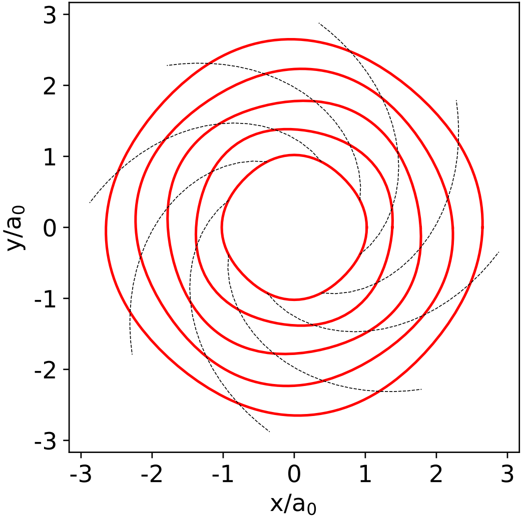


Example: Spiral Scaling FFA

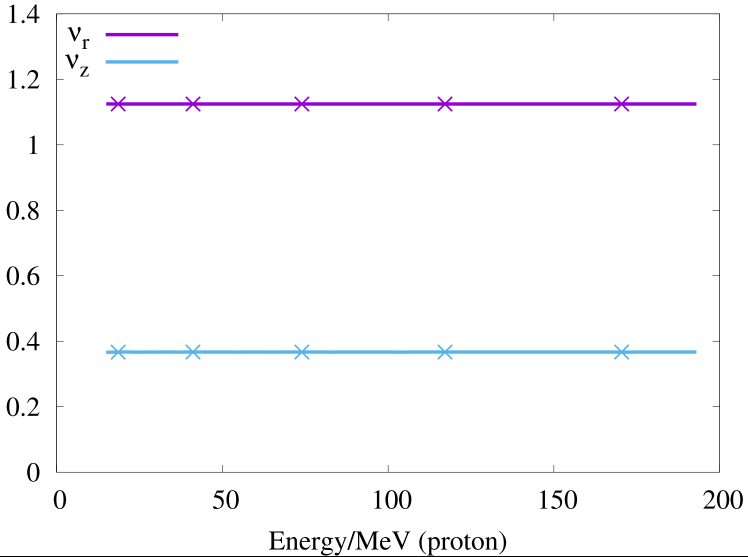
$$P(a) = P_0 \left(\frac{a}{a_0}\right)^{k+1}$$

$$r(a,\theta) = a \left(1 + C \cos\left(N\theta + N \tan(\zeta) \ln\left(\frac{a}{a_0}\right)\right)\right)$$

Example: Spiral Scaling FFA



Example: Spiral Scaling FFA



Conclusion:

- Test ideas: generate isochronous field maps in no time
- Parameterize a ring using only a few orbits: optimize
- New approach to designing FFA?
- It's not obvious how to arrange the steel to get the desired field…

Bibliography

Baartman, R. (2005).
Linearized Equations of Motion in Magnet with Median Plane Symmetry.
Technical Report TRI-DN-05-06, TRIUMF.

Courant, E. and Snyder, H. (1958).
Theory of the alternating-gradient synchrotron.
Annals of Physics, 3(1):1 – 48.

Gordon, M. (1968).
Orbit properties of the isochronous cyclotron ring with radial sectors.
Annals of Physics, 50(3):571–597.

Meade, C. (1971).
Changes to the main magnet code.
Technical Report TRI-DN-70-54-Addendum4, TRIUMF.